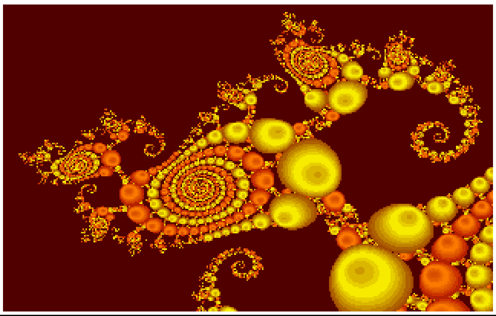
Complex number



Beautiful fractal diagram begins with a complex number.

Evaluate:

(1) $(1+i)^{2015}+(1-i)^{2015}$

(2) $1+(1+i)+(1+i)^2+(1+i)^3+\cdots+(1+i)^{2015}$.

Solve for all roots (including complex number roots):

(3) $z^6+z^3+1=0$

(4) $(z+1)^5+(z-1)^5=0$.

(1) Method 1

Since $-i(1+i)=1-i$,

$(1+i)^{2015}+(1-i)^{2015}=(1+i)^{2015}-i^{2015}(1+i)^{2015}$

$=(1+i)^{2015}-i^3(1+i)^{2015}=(1+i)^{2015}+i(1+i)^{2015}=(1+i)(1+i)^{2015}$

$=(1+i)^{2016}=(1+2i+i^2)^{1008}=(2i)^{1008}=2^{1008}$

Method 2 Note: $\text{cis}\,\theta=\cos\theta+i\sin\theta$

$$(1+i)^{2015}+(1-i)^{2015}=\left[\sqrt{2}\left(\text{cis}\,\frac{\pi}{4}\right)\right]^{2015}+\left[\sqrt{2}\left(\text{cis}\left(-\frac{\pi}{4}\right)\right)\right]^{2015}$$

$$=\sqrt{2}^{2015}\left[\text{cis}\left(\frac{\pi}{4}\times 2015\right)+\text{cis}\left(-\frac{\pi}{4}\times 2015\right)\right]$$, by de Morivre's Theorem

$$=\sqrt{2}^{2015}\left[2\cos\left(\frac{\pi}{4}\times 2015\right)\right]=\sqrt{2}(2^{1007})\left[2\cos\left(504\pi-\frac{\pi}{4}\right)\right]$$

$$=\sqrt{2}(2^{1008})\left[\cos\left(-\frac{\pi}{4}\right)\right]=\sqrt{2}(2^{1008})\left[\frac{1}{\sqrt{2}}\right]=2^{1008}$$

Method 3

Let $x=(1+i)^{2015}+(1-i)^{2015}$, $y=(1+i)^{2015}-(1-i)^{2015}$

$x+y=2(1+i)^{2015}=2(1+2i+i^2)^{1007}(1+i)=2(2i)^{1007}(1+i)$

$=2^{1008}(-i)(1+i)=2^{1008}(1-i)$ … (1)

$x-y=2(1-i)^{2015}=2(1-2i+i^2)^{1007}(1-i)=2(-2i)^{1007}(1-i)$

$=-2^{1008}(-i)(1-i)=2^{1008}(1+i)$ … (2)

$\frac{(1)+(2)}{2}$, $x=2^{1008}$. We also find $\frac{(1)-(2)}{2}$, $y=-2^{1008}i$.

(2) $1+(1+i)+(1+i)^2+(1+i)^3+\cdots+(1+i)^{2015}=\frac{(1+i)^{2016}-1}{(1+i)-1}=\frac{(1+2i+i^2)^{1008}-1}{i}$

$=(-i)[(2i)^{1008}-1]=(-i)[2^{1008}-1]=[1-2^{1008}]i$

(3) Method 1

Let $w=z^3$

$z^6+z^3+1=0 \Rightarrow w^2+w+1=0$

By quadratic eq. formula, $w=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i=\text{cis}\left(\pm\frac{2\pi}{3}\right)$

$z=w^{1/3}=\left[\text{cis}\left(\pm\frac{2\pi}{3}+2k\pi\right)\right]^{1/3}=\text{cis}\left(\pm\frac{2\pi}{9}+\frac{2k\pi}{3}\right)$, where k = 0, 1, 2.

When k = 0, $z=\cos\left(\frac{2\pi}{9}\right)\pm i\sin\left(\frac{2\pi}{9}\right)=0.766044443119\pm 0.6427876096865i$

When k = 1, $z=\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)=-0.9396926207859+0.3420201433257i$

$z=\cos\left(\frac{4\pi}{9}\right)+i\sin\left(\frac{4\pi}{9}\right)=0.1736481776669+0.9848077530122i$

When k = 2, $z=\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)=0.1736481776669-0.9848077530122i$

$z=\cos\left(\frac{10\pi}{9}\right)+i\sin\left(\frac{10\pi}{9}\right)=-0.9396926207859-0.3420201433257i$

Method 2

$z^6+z^3+1=\frac{z^9-1}{z^3-1}=0 \Rightarrow z^9-1=0$ and $z^3-1\neq 0$

For $z^9-1=0\Rightarrow z^9=1=\text{cis } 2k\pi \Rightarrow z=\text{cis }\frac{2k\pi}{9}$, k = 0, 1, 2, …, 8

For $z^3-1=0\Rightarrow z^3=1=\text{cis } 2k\pi \Rightarrow z=\text{cis }\frac{2k\pi}{3}$, k = 0, 1, 2.

Hence the roots are $z=\text{cis }\frac{2\pi}{9},\text{cis }\frac{6\pi}{9},\text{cis }\frac{8\pi}{9},\text{cis }\frac{10\pi}{9},\text{cis }\frac{14\pi}{9},\text{cis }\frac{16\pi}{9}$.

(4) Method 1

$$(z+1)^5+(z-1)^5=0 \Rightarrow (z+1)^5=-(z-1)^5 \Rightarrow \left(\frac{z+1}{z-1}\right)^5=-1=\text{cis}(2k\pi-\pi)$$

$\frac{z+1}{z-1}=\text{cis}\left(\frac{2k\pi-\pi}{5}\right)$, k = 0, 1, 2, 3, 4.

$z=\frac{\text{cis}\left(\frac{2k\pi-\pi}{5}\right)+1}{\text{cis}\left(\frac{2k\pi-\pi}{5}\right)-1}=\frac{\text{cis}\left(\frac{2k\pi-\pi}{10}\right)\left[\text{cis}\left(\frac{2k\pi-\pi}{10}\right)+\text{cis}\left(-\frac{2k\pi-\pi}{10}\right)\right]}{\text{cis}\left(\frac{2k\pi-\pi}{10}\right)\left[\text{cis}\left(\frac{2k\pi-\pi}{10}\right)-\text{cis}\left(-\frac{2k\pi-\pi}{10}\right)\right]}=\frac{2\cos\left(\frac{2k\pi-\pi}{10}\right)}{2i\sin\left(\frac{2k\pi-\pi}{10}\right)}=-\cot\left(\frac{2k\pi-\pi}{10}\right)i$

When k = 0, $z=-\cot\left(\frac{-\pi}{10}\right)\approx 3.0776835371753i$

When k = 1, $z = -\cot\left(\frac{\pi}{10}\right) \approx -3.0776835371753i$

When k = 2, $z = -\cot\left(\frac{3\pi}{10}\right) \approx -0.7265425280054i$

When k = 3, $z = -\cot\left(\frac{5\pi}{10}\right) = 0$

When k = 4, $z = -\cot\left(\frac{7\pi}{10}\right) \approx 0.7265425280054i$

Method 2

$(z+1)^5 + (z-1)^5 = 0 \Rightarrow z^5 + 10z^3 + 5z = 0 \Rightarrow z(z^4 + 10z^2 + 5) = 0$

Therefore $z = 0$ or $z^4 + 10z^2 + 5 = (z^2)^2 + 10z^2 + 5 = 0$

$$z^2 = 2\sqrt{5} - 5 \text{ or } z = -2\sqrt{5} - 5$$

$$z = \pm\sqrt{2\sqrt{5} - 5} \text{ or } \pm\sqrt{-2\sqrt{5} - 5}$$

$$z \approx \pm 0.7265425280054i \text{ or } \pm 3.0776835371753i$$

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